Chapter 6: AC Circuits

Chapter 6: Outline

AC Steady State response = Forced Response

↓

$$k_3 \cos \omega t + k_4 \sin \omega t \leftrightarrow K_3 \cos \omega t + K_4 \sin \omega t$$

$$X_m \cos(\omega t + \phi) \leftrightarrow X'_m \cos(\omega t + \phi')$$

Phasor representation

$$\underline{X} = X_m \angle \phi \leftrightarrow \underline{X'} = X'_m \angle \phi'$$

↓

With Phasor Notations, circuit equations become algebraic.
Thus, all resistive circuit analysis methods are applicable.

$$i = C \frac{dv}{dt} \rightarrow \underline{I} = j\omega C \underline{V}$$

↓

Resistance → Impedance

Conductance → Admittance

Complex, frequency dependence

Phasors and the AC Steady State

AC Circuits

- A stable, linear circuit operating in the steady state with sinusoidal excitation (i.e., sinusoidal steady state).
- Complete response = forced response + natural response.
- In the steady state, natural response → 0.

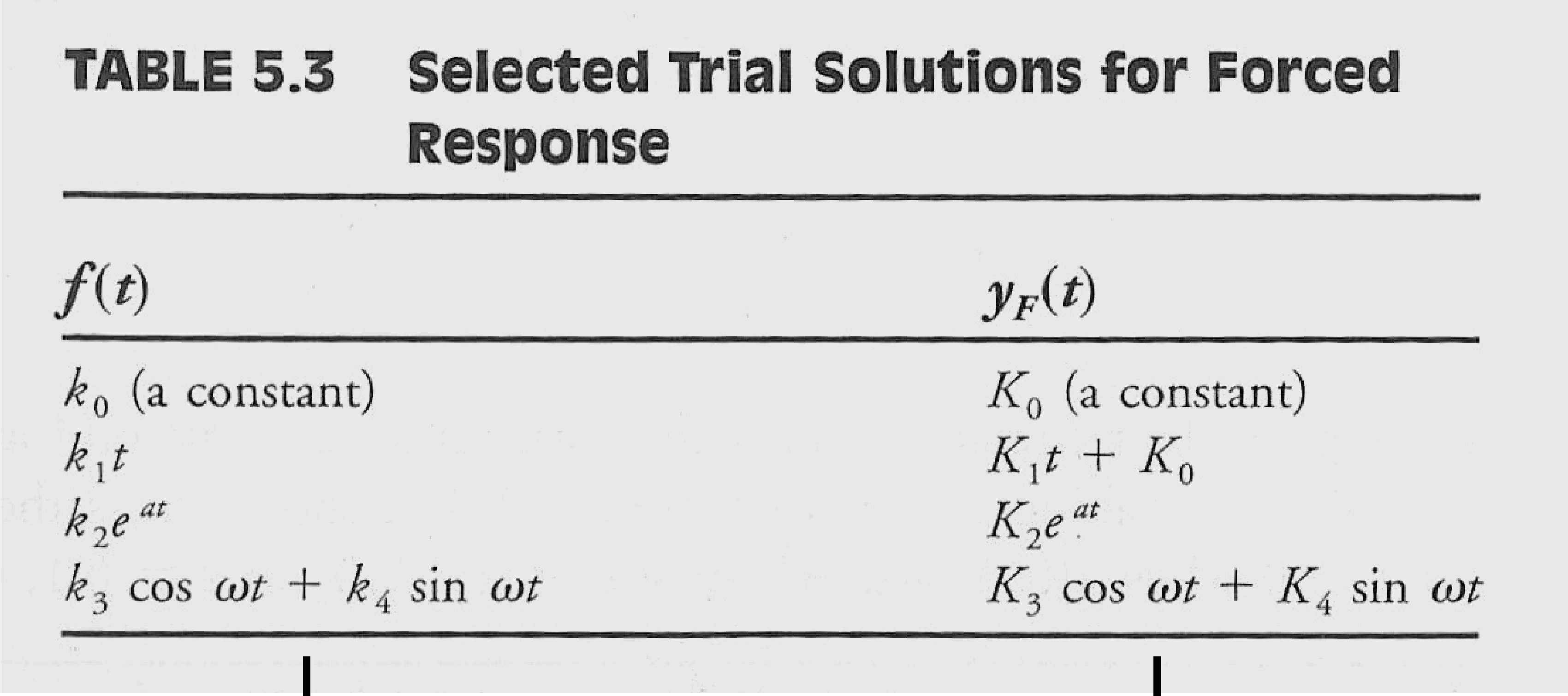


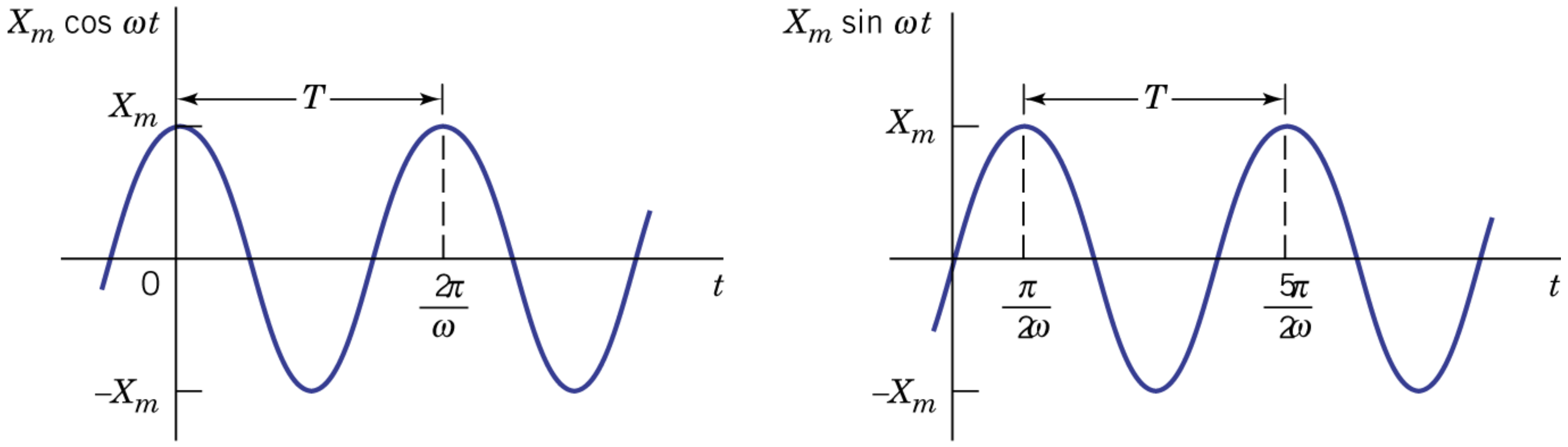
TABLE 5.3 Selected Trial Solutions for Forced Response

$f(t)$	$y_F(t)$
k_0 (a constant)	K_0 (a constant)
$k_1 t$	$K_1 t + K_0$
$k_2 e^{at}$	$K_2 e^{at}$
$k_3 \cos \omega t + k_4 \sin \omega t$	$K_3 \cos \omega t + K_4 \sin \omega t$

$$A_1 \cos(wt + q_1) \qquad A_2 \cos(wt + q_2)$$

Same frequency, different amplitudes and phases

Sinusoids

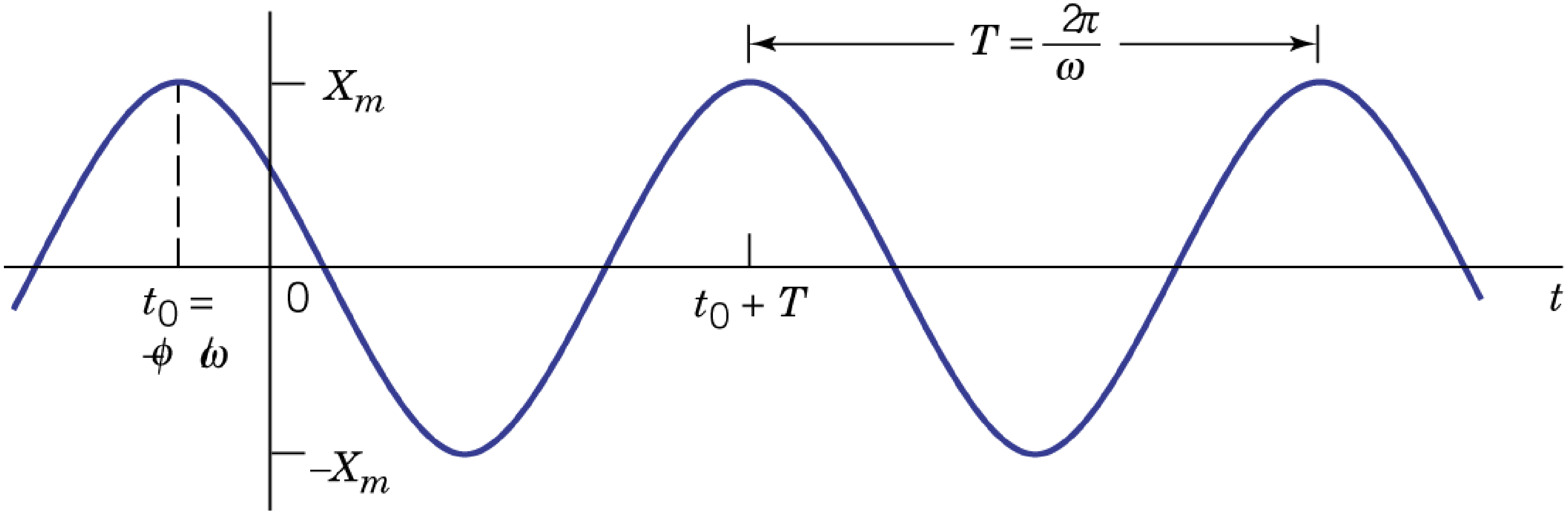


(a) Cosine wave

(b) Sine wave

- Three parameters are needed to determine a sinusoid.
- $x(t)=X_m cos(wt+f)=Re[X_m e^{j(wt+f)}]$.
- X_m: amplitude, $w=2pf=2p/T$: angular frequency, f: phase angle (radian).

Phasors



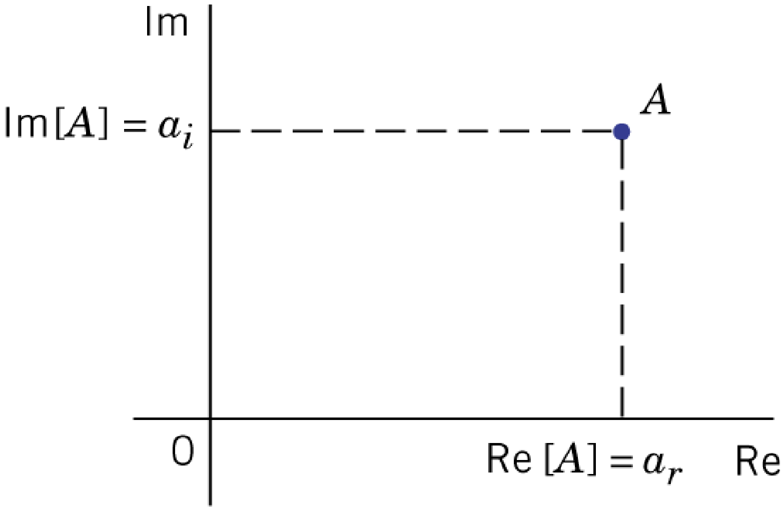
- The three parameters can be represented by a rotating phasor in a two-dimensional plane.
- At a given time (e.g., $t=0$), the nonrotating phasor is represented by $\underline{X} = X_m \angle f$.
- The frequency information is not included.

AC Forced Response

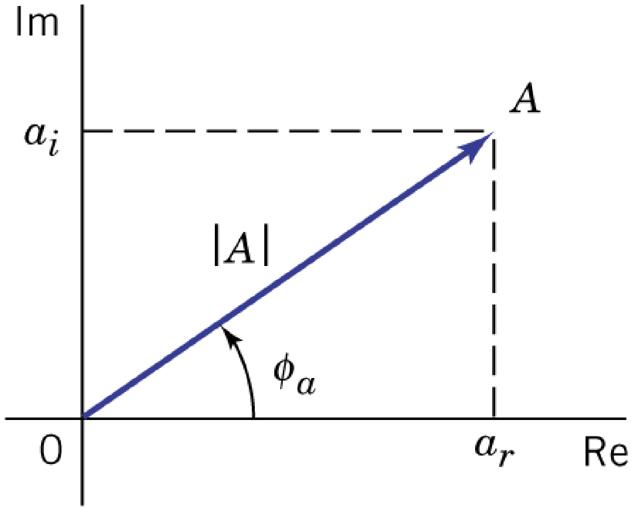
- The forced response of any branch variable (current or voltage) is at the same frequency as the excitation frequency w for a linear circuit.
- In other words, any branch variable has the general form $y(t)=Y_m cos(wt+f_y)$.
- Circuit analysis becomes manipulation of complex numbers.

Complex Numbers in the Complex Plane

$$j \equiv \sqrt{-1}$$



(*a*) The complex plane with
$A = a_r + ja_i$



(*b*) Polar coordinates for
$A = |A| \underline{/\phi_a}$

$$a_r = |A| \cos f_a$$

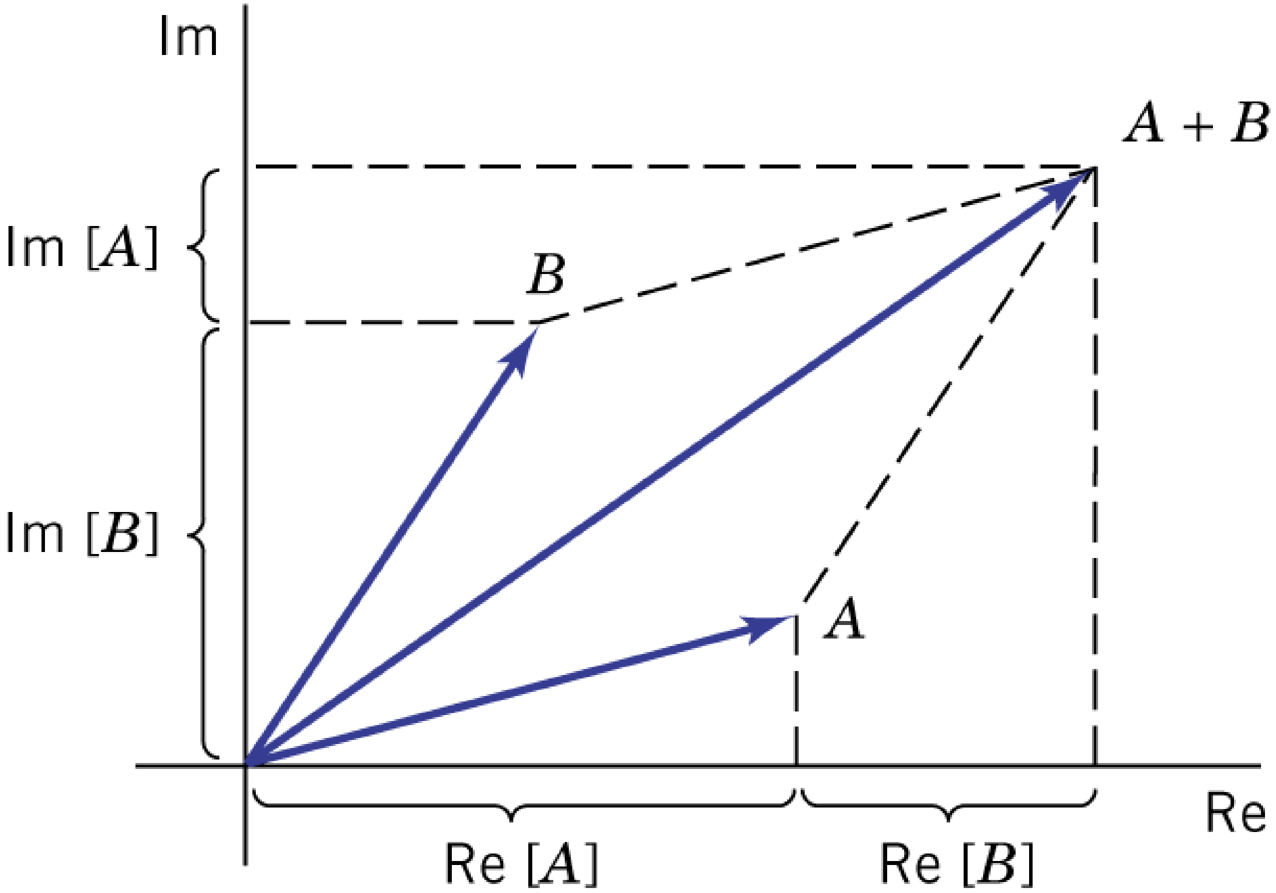
$$a_i = |A| \sin f_a$$

$$|A| = \left(a_r^2 + a_i^2\right)^{1/2}$$

$$f_a = \tan^{-1}\left(\frac{a_i}{a_r}\right) \quad a_r > 0$$

$$f_a = \pm 180^0 - \tan^{-1}\left(\frac{a_i}{a}\right) \quad a_r < 0$$

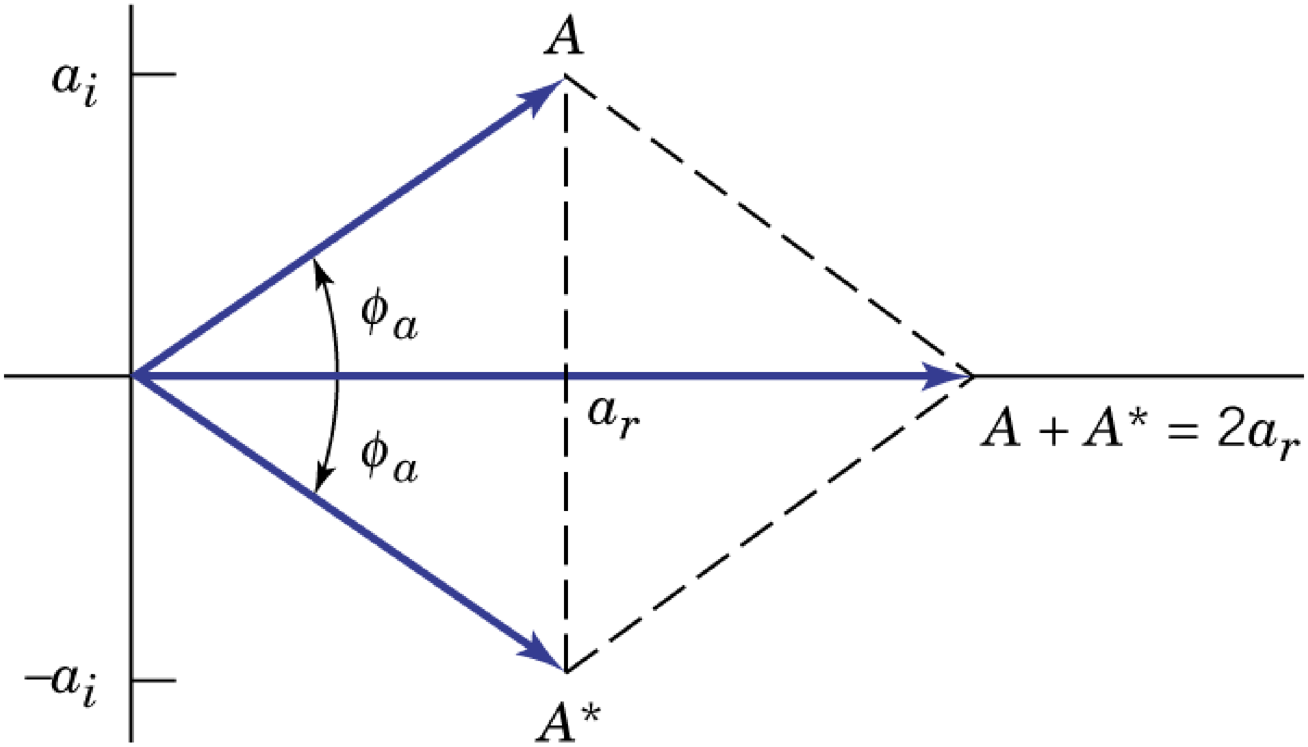
Complex Addition and Subtraction



$$\mathrm{Re}\left[A \pm B\right] = \mathrm{Re}\left[A\right] \pm \mathrm{Re}\left[B\right]$$

$$\mathrm{Im}\left[A \pm B\right] = \mathrm{Im}\left[A\right] \pm \mathrm{Im}\left[B\right]$$

Complex Conjugate



$$A = |A| \angle f_a = a_r + j \cdot a_i$$

$$A^* \equiv |A| \angle - f_a = a_r - j \cdot a_i$$

$$A \cdot A^* = a_r^2 + a_i^2 = |A|^2$$

Complex Multiplication

$$A \cdot B = (a_r b_r - a_i b_i) + j \cdot (a_r b_i - a_i b_r) = \mathrm{Re}\left[AB\right] + \mathrm{Im}\left[AB\right]$$

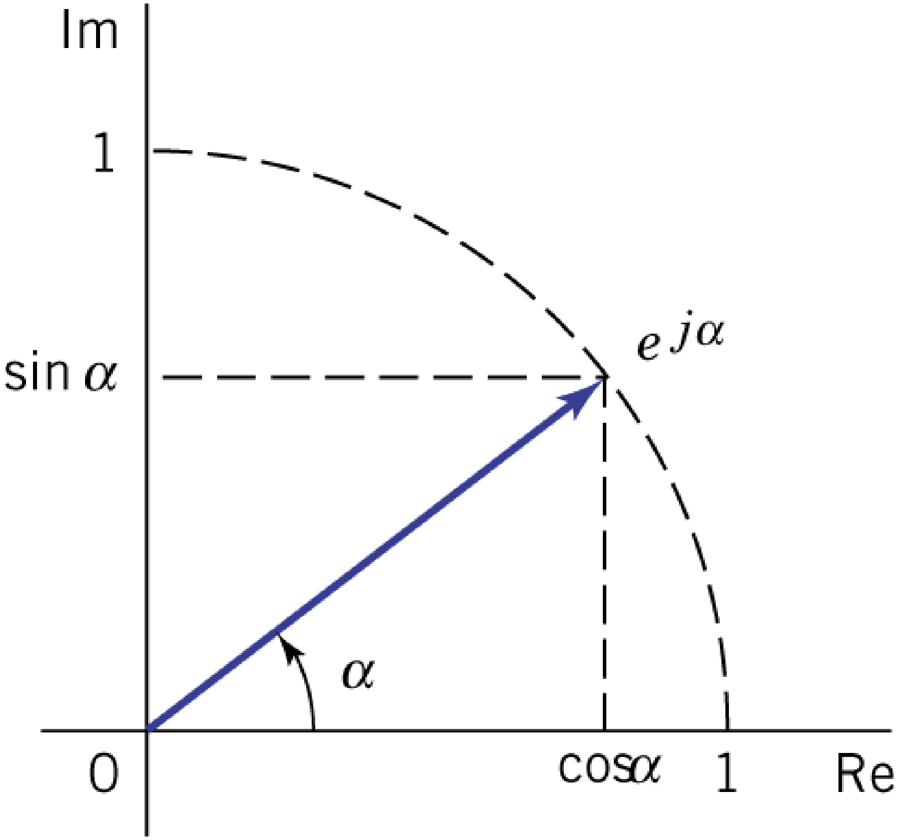
$$\text{if } k \text{ is real}, \mathrm{Re}[kB] = k\,\mathrm{Re}[B], \mathrm{Im}[kB] = k\,\mathrm{Im}[B]$$

Complex Division (Rationalization)

$$\frac{B}{A} = \frac{BA^*}{AA^*} = \frac{b_r a_r + b_i a_i}{a_r^2 + a_i^2} + j\frac{b_r a_i - b_i a_r}{a_r^2 + a_i^2}$$

Complex Number in Exponential Form

- Euler’s formula: $e^{\pm ja} = \cos a \pm j \sin a$
- Complex number in exponential form: $A = |A| e^{jf_a}$



$$A \cdot B = |A||B| \angle f_a + f_b$$

$$\frac{A}{B} = \frac{|A|}{|B|} \angle f_a - f_b$$

$$j = 1 \angle 90^0$$

Phasor Representation

- A sinusoid can be represented by a phasor:

$$X\cos(wt+f)=\mathrm{Re}\left[Xe^{jwt+f}\right]=\mathrm{Re}\left[Xe^{jf}e^{jwt}\right]=\mathrm{Re}\left[\underline{X}e^{jwt}\right]$$

- The sum of two sinusoids at the same frequency can be represented by another phasor. The new phasor is simply the sum of the two original phasors.

$$\mathrm{Re}\left[\underline{X}_1e^{jwt}\right]+\mathrm{Re}\left[\underline{X}_2e^{jwt}\right]=\mathrm{Re}\left[\underline{X}_1e^{jwt}+\underline{X}_2e^{jwt}\right]=\mathrm{Re}\left[(\underline{X}_1+\underline{X}_2)e^{jwt}\right]$$

Phasor Representation

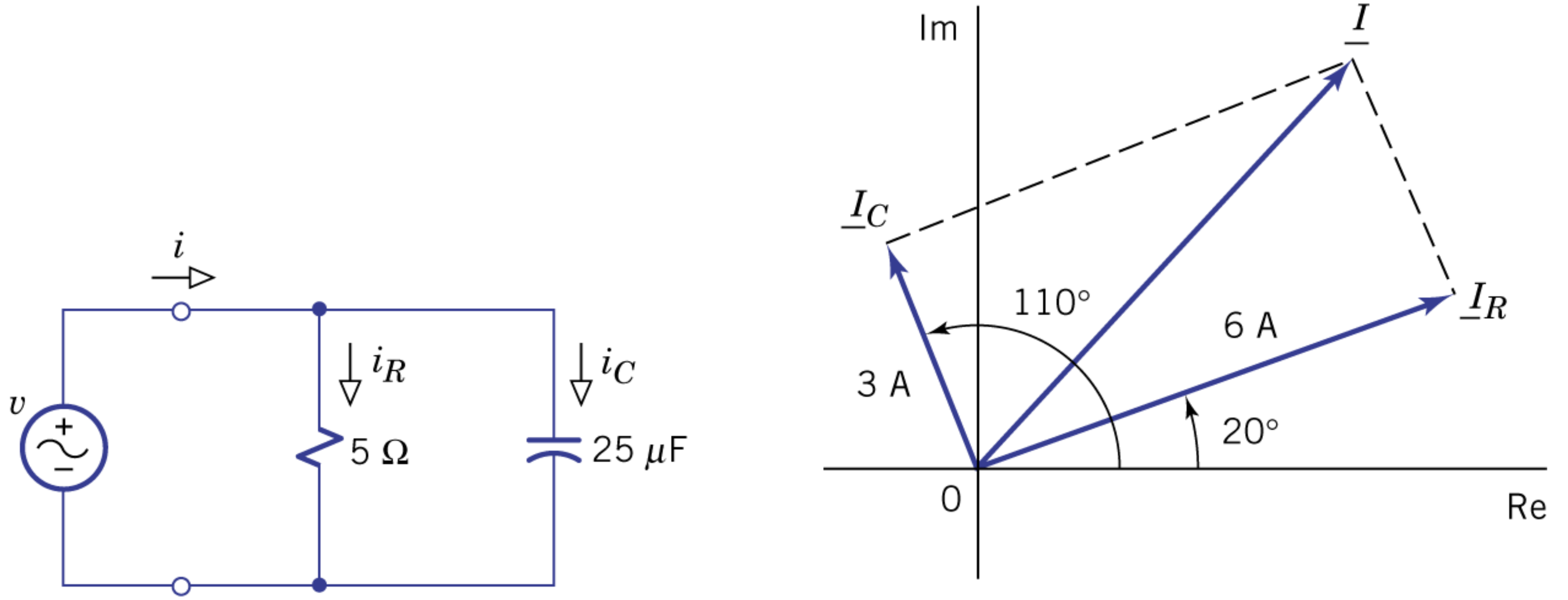
- The steady-state response of any branch variable in a stable circuit with a sinusoidal excitation will be another sinusoid at the same frequency (forced response in Chapter 5)
- Kirchhoff’s laws hold in phasor form (only additions are involved).

Phasor with Differentiation

$$\frac{d\,\mathrm{Re}\left[Xe^{jwt+f}\right]}{dt} = \mathrm{Re}\left[\frac{dXe^{jwt+f}}{dt}\right] = \mathrm{Re}\left[jwXe^{jwt+f}\right]$$

$$\underline{X} \xrightarrow{differentiation} jw\,\underline{X}$$

Example 6.3: Parallel Network with an AC Voltage Source



(a) ***RC*** circuit in the ac steady state

(b) Phasor diagram for the currents

$$v(t) = 30\cos(4000t + 20^0) \Rightarrow \underline{V} = 30\angle 20^0$$

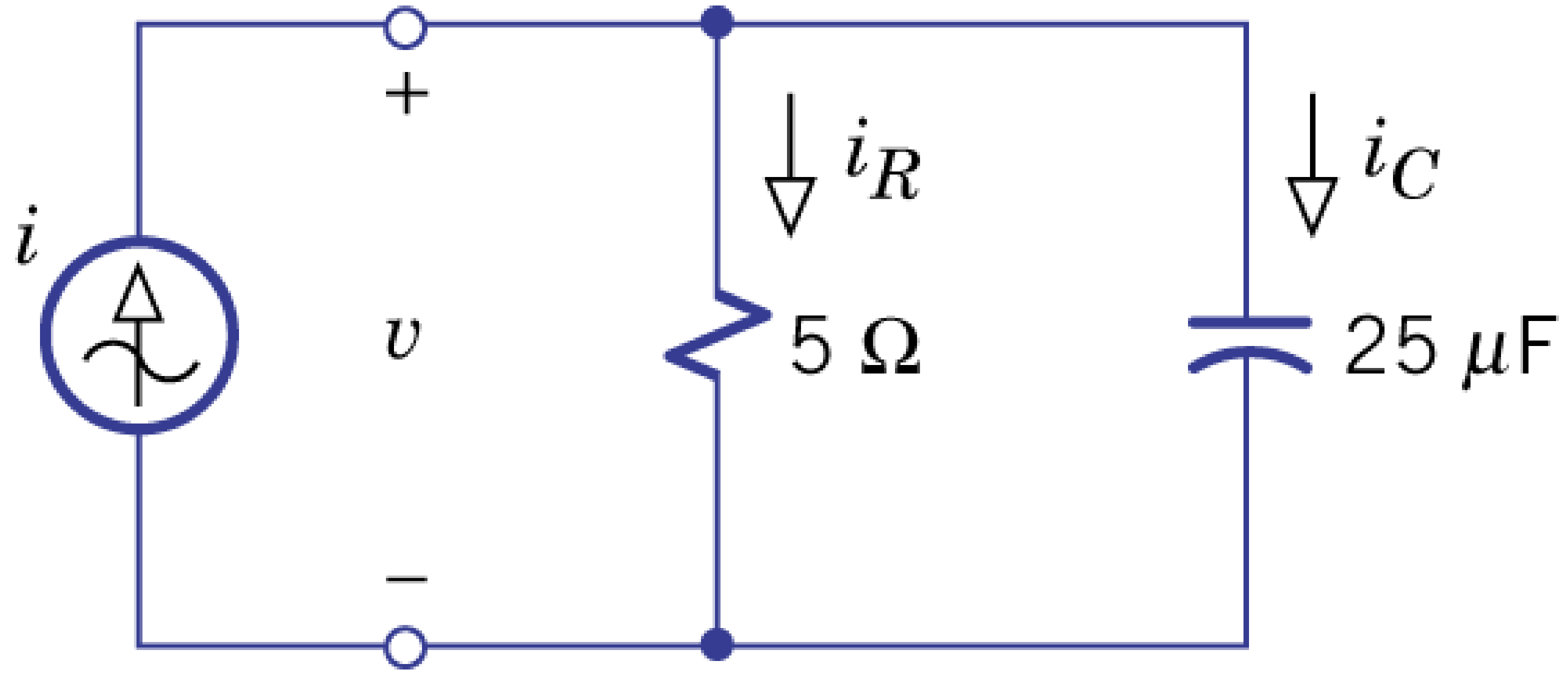
$$i_R = \frac{v}{5} \Rightarrow \underline{I}_R = \frac{\underline{V}}{5}$$

$$i_C(t) = C\frac{dv_c(t)}{dt} \Rightarrow \underline{I}_C = C \cdot jw\underline{V}$$

$$\underline{I} = \underline{I}_R + \underline{I}_C = 6.71\angle 46.6(A)$$

$$i(t) = 6.71\cos(4000t + 46.6^0)(A)$$

Example 6.4: Parallel Network with an AC Current Source



$$i(t) = 3\cos 4000t \Rightarrow \underline{I} = 3$$

$$\underline{I}_R = \frac{\underline{V}}{5}, \underline{I}_C = C \cdot jw\underline{V}$$

$$\underline{I} = 3 = \underline{I}_R + \underline{I}_C = (0.2 + j \cdot 0.1)\underline{V}$$

$$\underline{V} = \frac{3}{0.2 + j \cdot 0.1}$$

$$v(t) = 13.4\cos(4000t - 26.6^0)(V)$$

Impedance and Admittance

Phasor Representation

- Under ac steady-state, both the voltage and the current of a branch are sinusoids at the same frequency.

$$v(t) = V_m \cos(wt + f_v) = \mathrm{Re}\left[\underline{V} e^{jwt}\right]$$

$$i(t) = I_m \cos(wt + f_i) = \mathrm{Re}\left[\underline{I} e^{jwt}\right]$$

Resistors

- Current and voltage are collinear (in phase).

$$v = \mathrm{Re}\left[\underline{V}e^{jwt}\right] = R \times \mathrm{Re}\left[\underline{I}e^{jwt}\right] = \mathrm{Re}\left[R\underline{I}e^{jwt}\right]$$

$$\underline{V} = R\underline{I} = V_m \angle f_v = RI_m \angle f_i$$

$$V_m = RI_m, f_v = f_i$$

Inductors

- Current lags voltage by 90 degrees.

$$v = L\frac{di}{dt} = L\frac{d}{dt}\mathrm{Re}\left[\underline{I}e^{jwt}\right] = L \times \mathrm{Re}\left[\underline{I}\frac{de^{jwt}}{dt}\right] = \mathrm{Re}\left[jwL\underline{I}e^{jwt}\right] = \mathrm{Re}\left[\underline{V}e^{jwt}\right]$$

$$\underline{V} = jwL\underline{I}$$

$$V_m = wLI_m, f_v = f_i + 90^{\mathrm{o}}$$

Capacitors

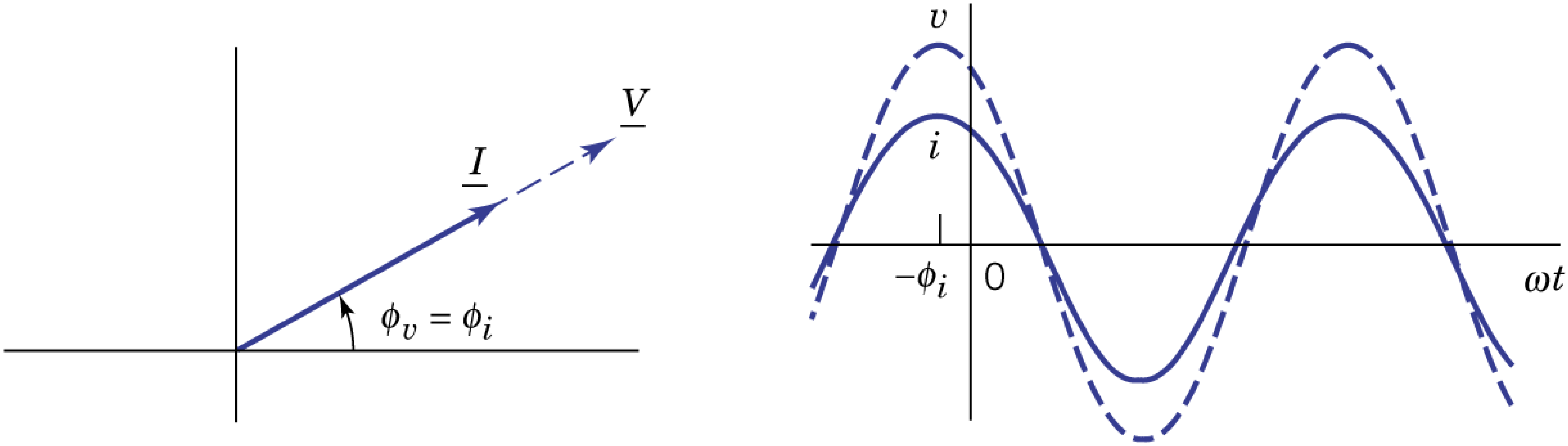
- Current leads voltage by 90 degrees.

$$i = C\frac{dv}{dt} = C\frac{d}{dt}\mathrm{Re}\left[\underline{V}e^{jwt}\right] = C\times\mathrm{Re}\left[\underline{V}\frac{de^{jwt}}{dt}\right] = \mathrm{Re}\left[jwC\underline{V}e^{jwt}\right] = \mathrm{Re}\left[\underline{I}e^{jwt}\right]$$

$$\underline{V} = \frac{1}{jwC}\underline{I} = -\frac{j}{wC}\underline{I}$$

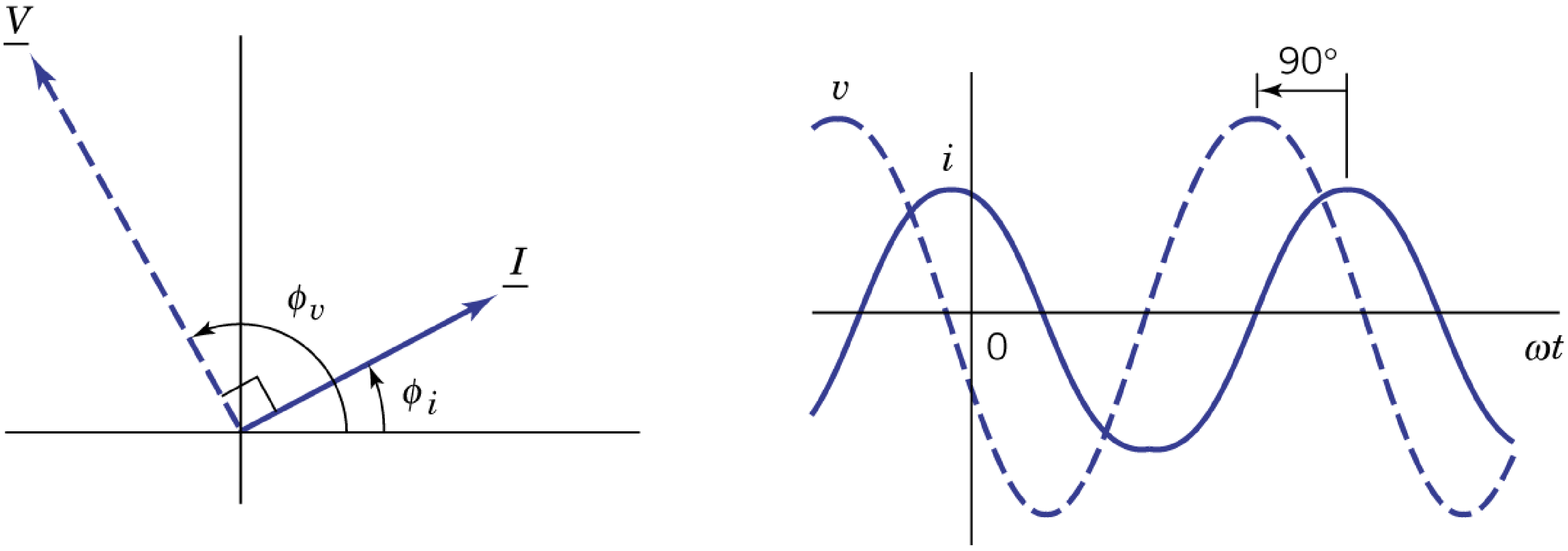
$$V_m = \frac{I_m}{wC}, f_v = f_i - 90^{o}$$

Phasor Relations (Resistor)



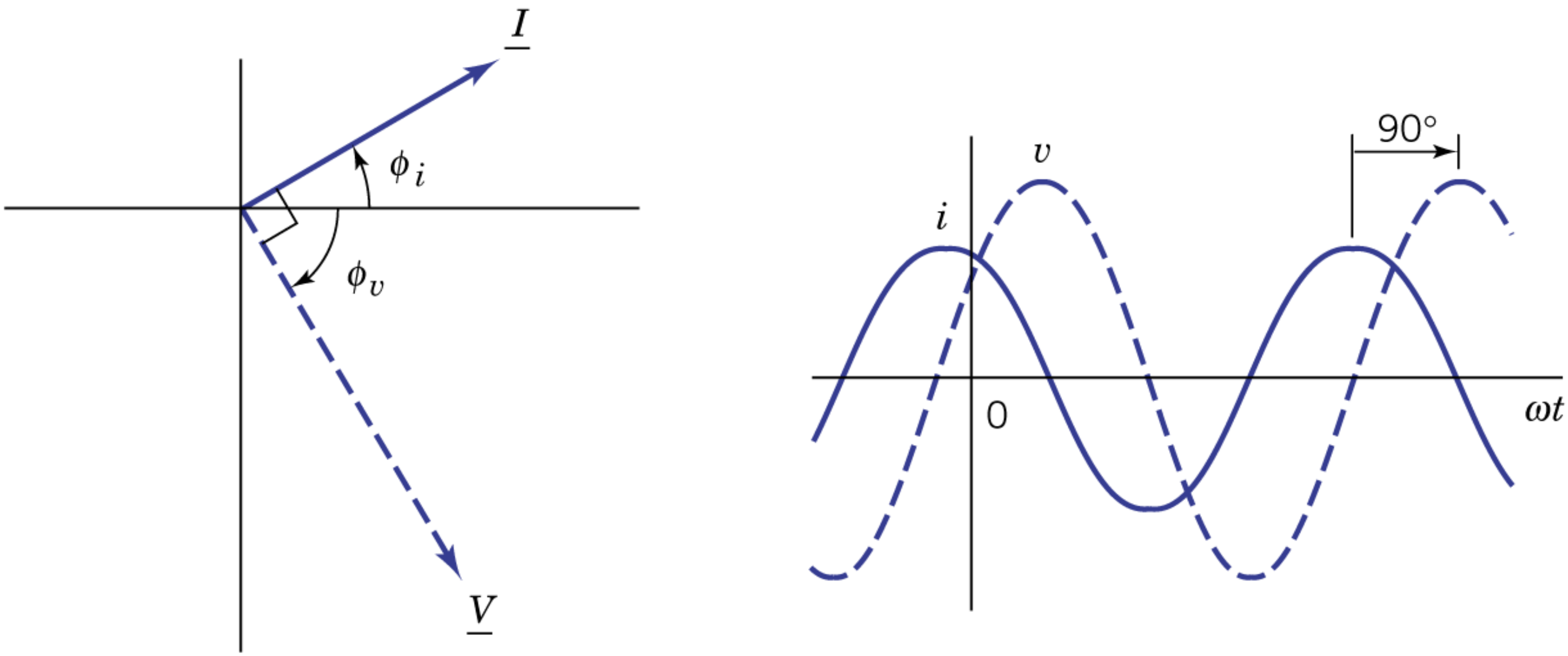
(a) Resistor

Phasor Relations (Inductor)



(b) Inductor

Phasor Relation (Capacitor)



(c) Capacitor

Impedance

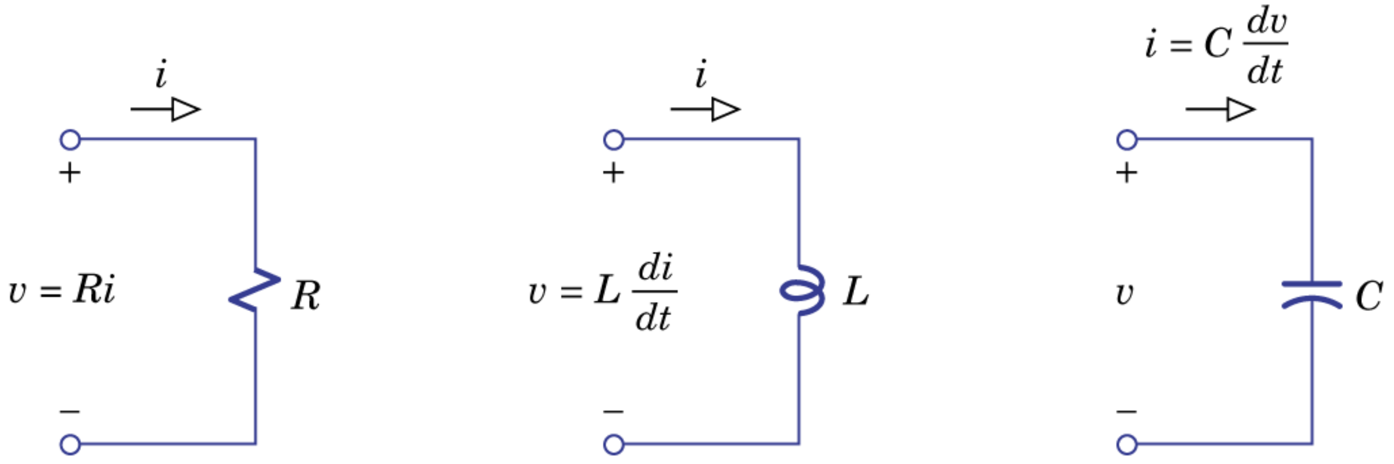
- In general, we can define a quantity Z and Ohm's law for ac circuits as $\underline{V} = Z\underline{I}$

$$Z_R = R$$

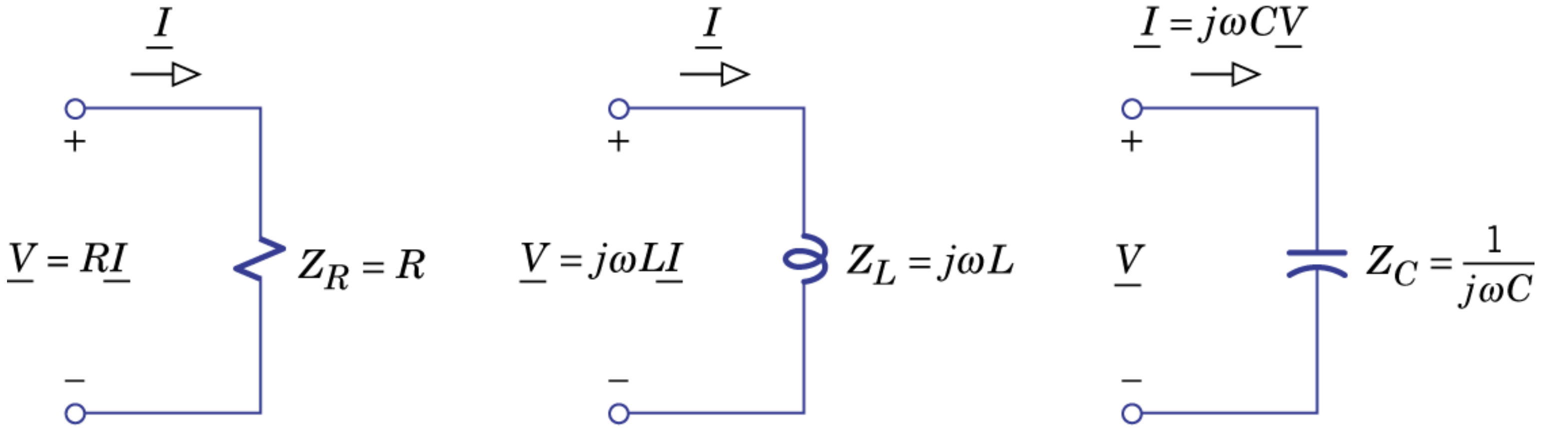
$$Z_L = jwL = wL\angle 90^{o}$$

$$Z_c = 1/jwC = 1/wC\angle -90^{o}$$

Time Domain vs. Frequency Domain



(a) Elements in the time domain



(b) Frequency-domain diagrams

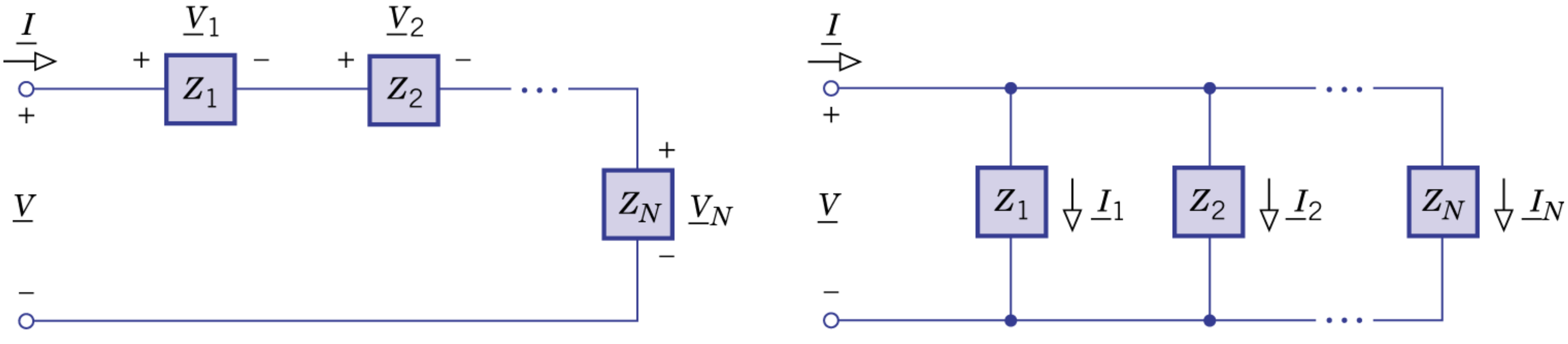
Admittance

- Similarly, another quantity admittance *Y* can be defined.

$$Y \equiv 1/Z$$

$$\underline{I} = Y\underline{V}$$

Equivalent Impedance and Admittance



(a) Impedances in series (b) Impedances in parallel

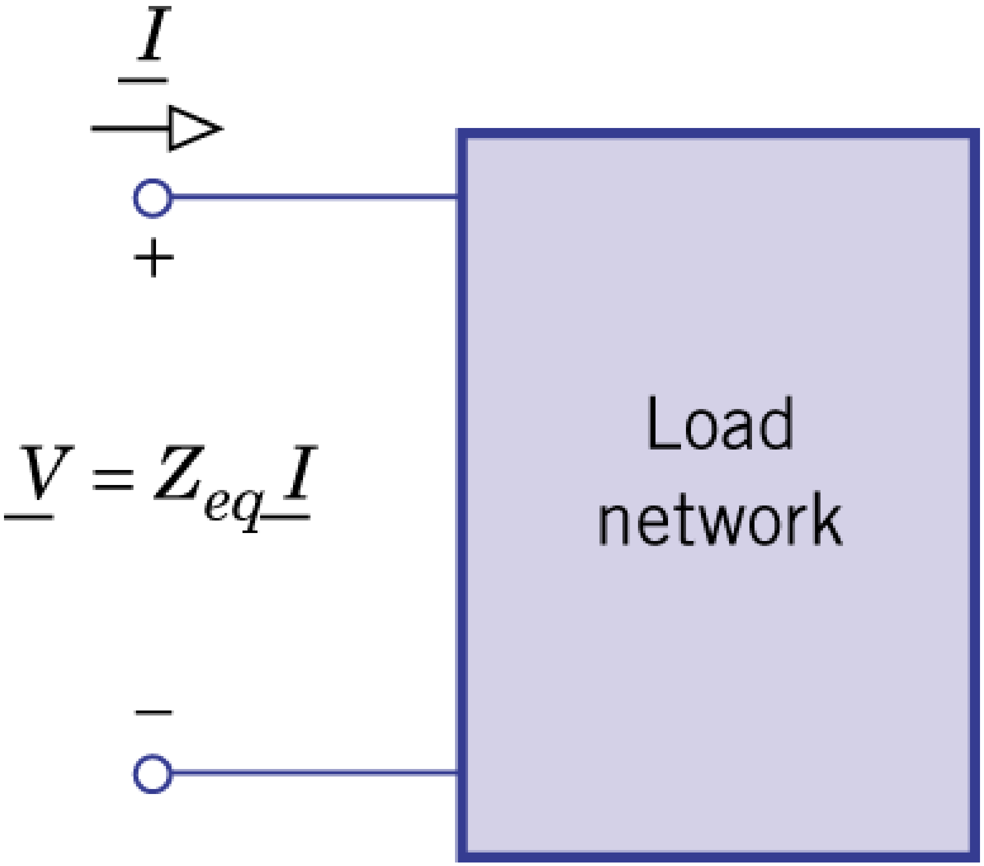
- Series equivalent impedance: $Z_{ser} = Z_1 + Z_2 + \cdots + Z_N$

  $(\underline{V} = \underline{V}_1 + \underline{V}_2 + \cdots + \underline{V}_N)$

- Parallel equivalent impedance: $Y_{par} = Y_1 + Y_2 + \cdots + Y_N$

  $(\underline{I} = \underline{I}_1 + \underline{I}_2 + \cdots + \underline{I}_N)$

Load Network



$$\underline{V} = Z_{eq}\underline{I}$$

$$Z_{eq} \equiv \underline{V} / \underline{I}$$

Impedance and Admittance

- Impedance and admittance are complex functions of frequency.

$$Z = Z(jw) = \mathrm{Re}[Z] + j\,\mathrm{Im}[Z] = \underset{\substack{\uparrow \\ \text{Resistance }(\Omega)}}{R(w)} + j\underset{\substack{\nwarrow \\ \text{Reactance }(\Omega)}}{X(w)}$$

- Inductors and capacitors are reactive elements, inductive reactance is positive and capacitive reactance is negative.

Impedance and Admittance

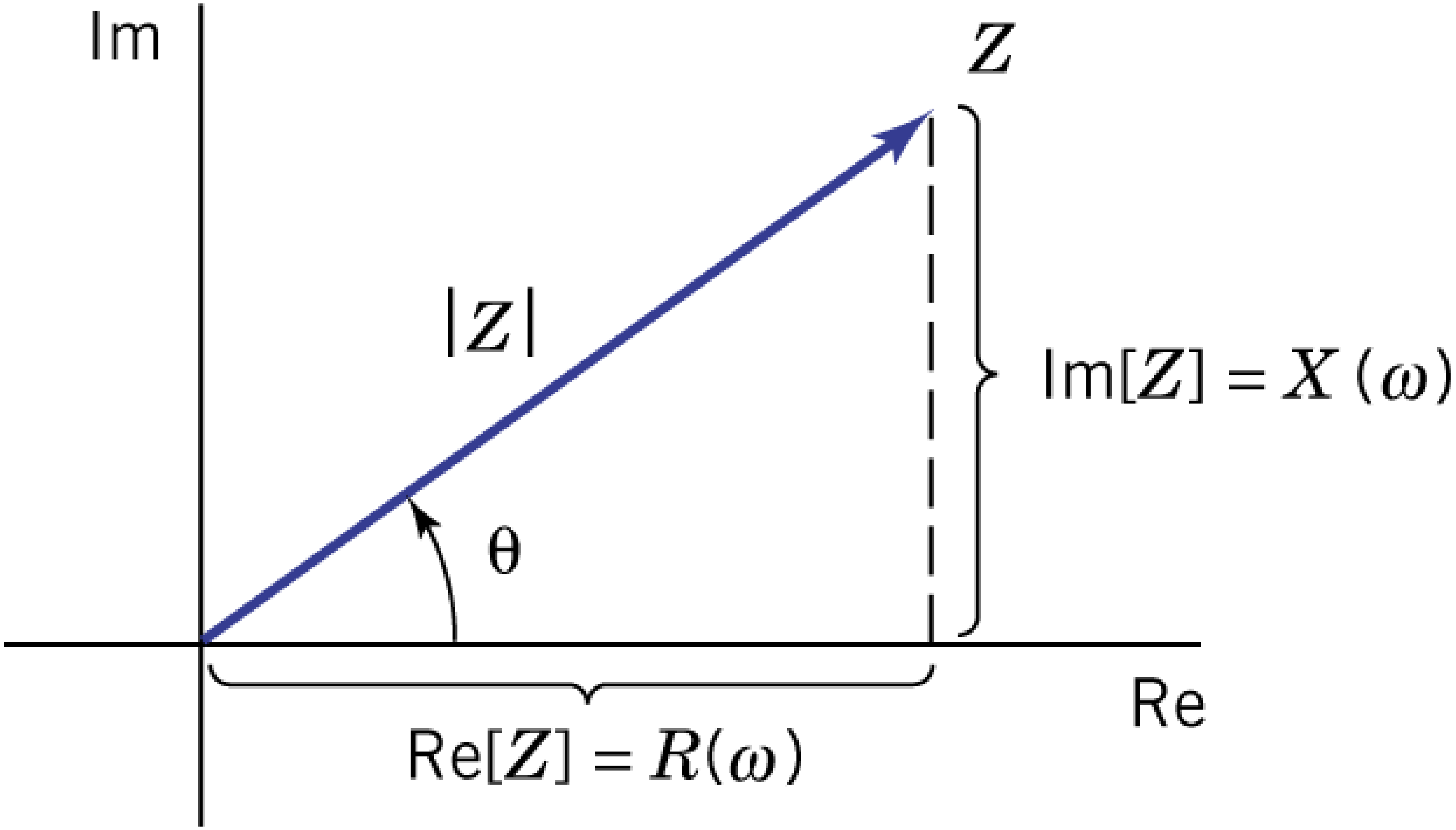
$$Y = Y(jw) = \mathrm{Re}[Y] + j\,\mathrm{Im}[Y] = G(w) + jB(w)$$

Conductance
(Siemens)

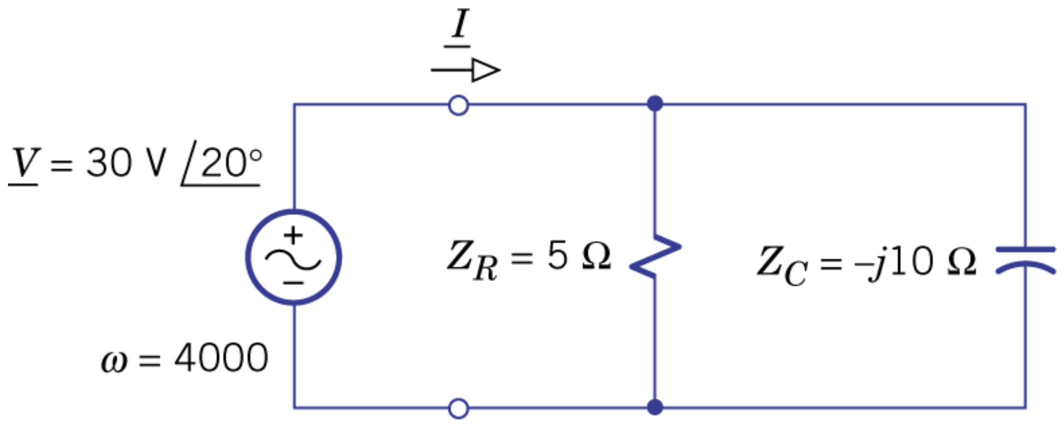
Susceptance
(Siemens)

- Inductors and capacitors are reactive elements, inductive reactance is positive and capacitive reactance is negative.

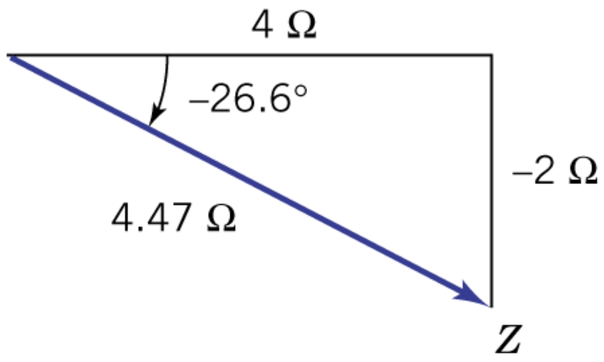
Impedance Triangle



Example 6.6: Impedance Analysis of a Parallel RC Circuit.



(*a*) Frequency-domain circuit diagram



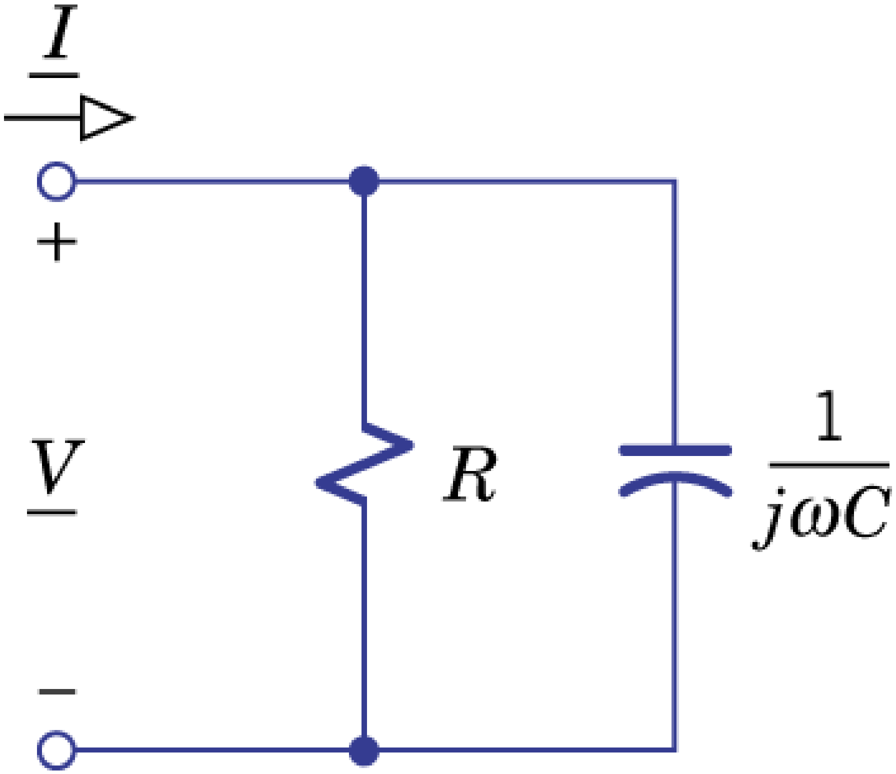
(*b*) Impedance triangle

$$Z = 5 \parallel -j10 = 4.47\Omega\angle -26.6^0$$

$$Y = \frac{1}{5} + \frac{1}{-j10} = 0.2 + j0.15 = 0.224S\angle 26.6^0$$

$$\underline{I} = \frac{\underline{V}}{\underline{Z}} = \underline{V}Y = 6.71A\angle 46.6^0$$

Example 6.7: Frequency Dependence of a Parallel RC Network.



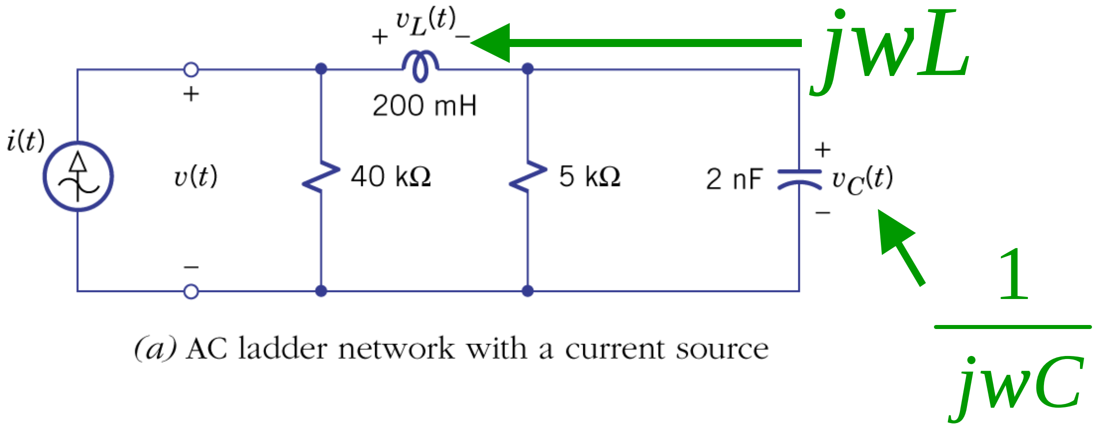
$$Z(jw) = \frac{Z_R Z_C}{Z_R + Z_C} = \frac{R - jwCR^2}{1 + (wCR)^2}$$

$$R(w) = \mathrm{Re}[Z] = \frac{R}{1 + (wCR)^2}$$

$$X(w) = \mathrm{Im}[Z] = -\frac{wCR^2}{1 + (wCR)^2}$$

Example 6.8: AC Ladder Calculations

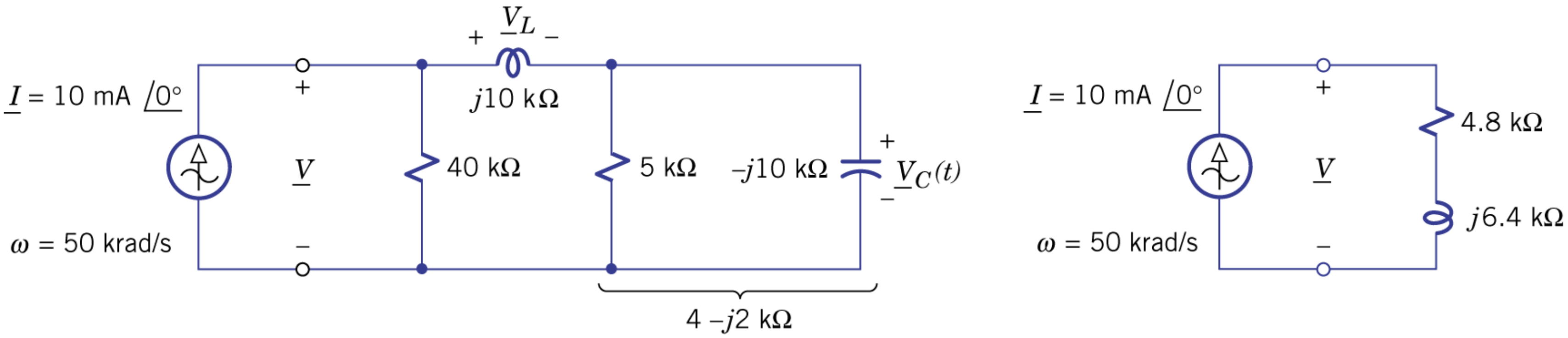
- AC ladder networks can be analyzed by series-parallel reduction (by replacing resistance with impedance).



jwL

$\frac{1}{jwC}$

(a) AC ladder network with a current source



(b) Frequency-domain diagram

(c) Equivalent impedances

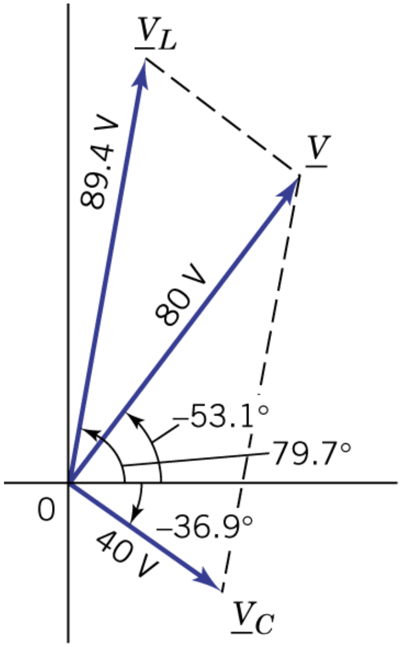
$$i(t)=10\cos 50000t(mA)$$

Example 6.8: (Cont.)

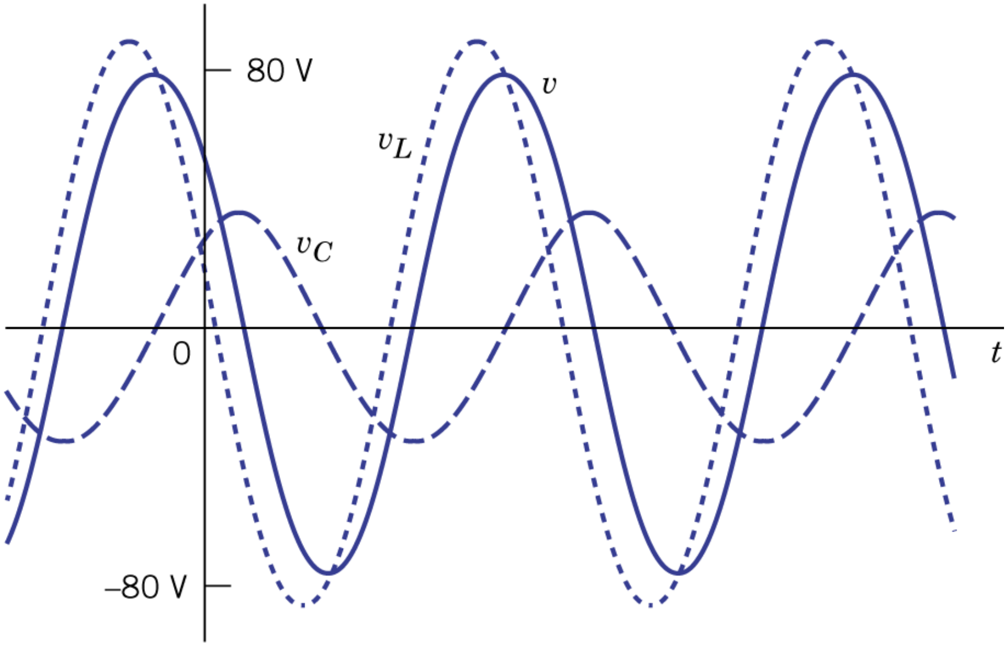
$$\underline{V} = Z\,\underline{I} = 48 + j64 = 80V\angle 53.1^0$$

$$\underline{V}_L = \frac{j10}{(4 - j2) + j10}\underline{V} = 89.4V\angle 79.7^0$$

$$\underline{V}_C = \frac{4 - j2}{(4 - j2) + j10}\underline{V} = 40V\angle -36.9^0$$



(a) Voltage phasors



(b) Voltage waveforms

AC Circuit Analysis

AC Circuit Analysis

- Sources at the same frequency:
  - Phasor transform method: the time domain sinusoids are transformed to the frequency domain and represented by phasors.

Time domain → *Frequency domain*

AC Circuit Analysis

- Sources at the same frequency:
  - With the transformation, all resistive circuit analysis techniques are applicable. Resistance is replaced by impedance and conductance is replaced by admittance.

Proportionality
Thévenin-Norton
Node Analysis
Mesh Analysis
…

AC Circuit Analysis

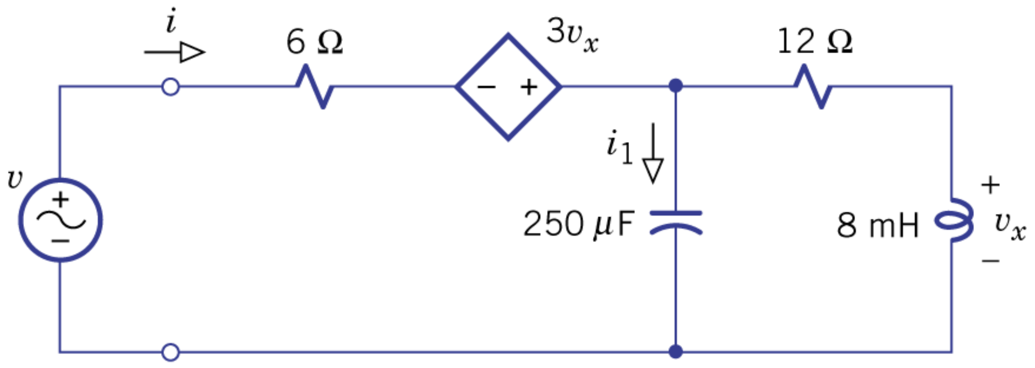
- Sources at the same frequency:
  - After analysis, the resultant phasors are converted back to the time-domain sinusoids.

Frequency domain → *Time domain*

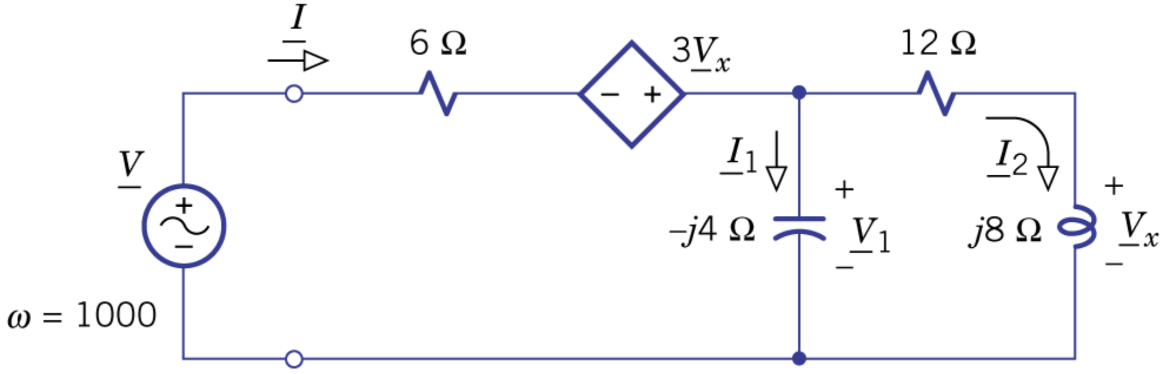
AC Circuit Analysis

- Sources at different frequencies:
  - Due to the linearity, the proportionality method is still applicable.
  - The phasor analysis is performed at each individual frequency

Example 6.9: AC Network with a Controlled Source



(a) AC network with a controlled source



(b) Frequency-domain diagram

$$\text{Let } \underline{I}_2 = 1\angle 0^0 (= 1 + j0)$$

$$\underline{V}_1 = 12 + j8$$

$$\underline{I}_1 = \underline{V}_1 / (-j4) = -2 + j3$$

$$\underline{I} = \underline{I}_1 + \underline{I}_2 = -1 + j3$$

$$\underline{V} = 6\underline{I} - 3\underline{V}_x + \underline{V}_1 = 6 + j2$$

$$Z = \underline{V} / \underline{I} = -j2$$

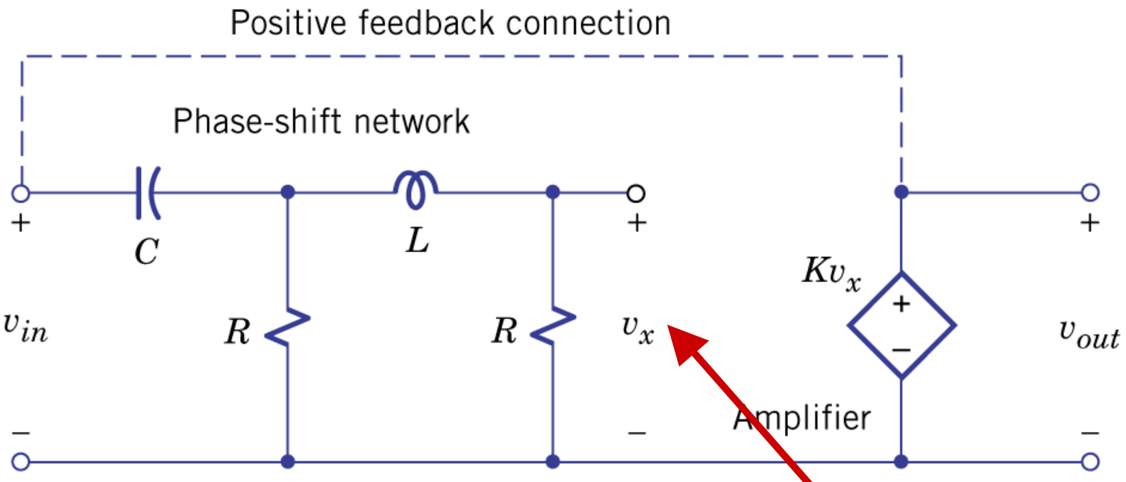
$$\underline{I} = \underline{V} / Z = 10A\angle 90^0$$

$$\underline{I}_1 = (\underline{I}_1 / \underline{I})\underline{I} = 11.4A\angle 105.3^0$$

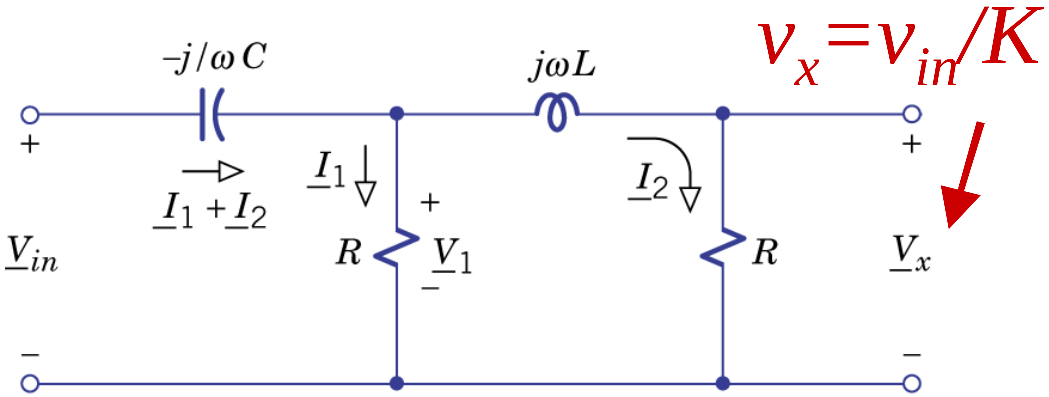
$$i_1 = 11.4\cos(1000t + 105.3^0)A$$

$v = 20\cos 1000t(V)$ Use proportionality

Example 6.10: Phase-Sift Oscillator



(*a*) Phase-shift oscillator circuit

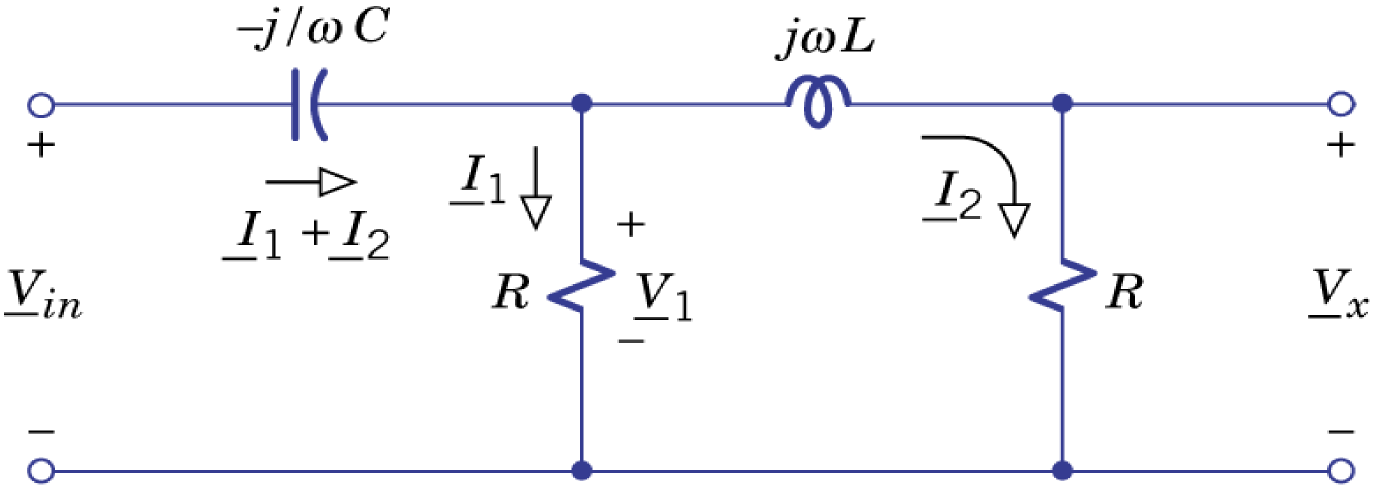


$v_x=v_{in}/K$

(*b*) Phase-shift network in the frequency domain

- Oscillator: Generator a sinusoidal output without an independent input source with initial stored energy.
- Design goal: At one particular frequency, $v_{out}=v_{in}$.

Example 6.10: (Cont.)



(b) Phase-shift network in the frequency domain

Use proportionality and let $\underline{I}_2 = 1$,

$$\frac{\underline{V}_{in}}{\underline{V}_x} = 1 + \left(\frac{L}{CR^2}\right) + j\left(wL - \frac{2}{wC}\right) / R$$

Oscillation requires : $wL = 2 / wC$

$$w_{OSC} = \sqrt{2 / LC}$$

when $w = w_{OSC}, \frac{\underline{V}_{in}}{\underline{V}_x} = K = 1 + \left(\frac{L}{CR^2}\right)$

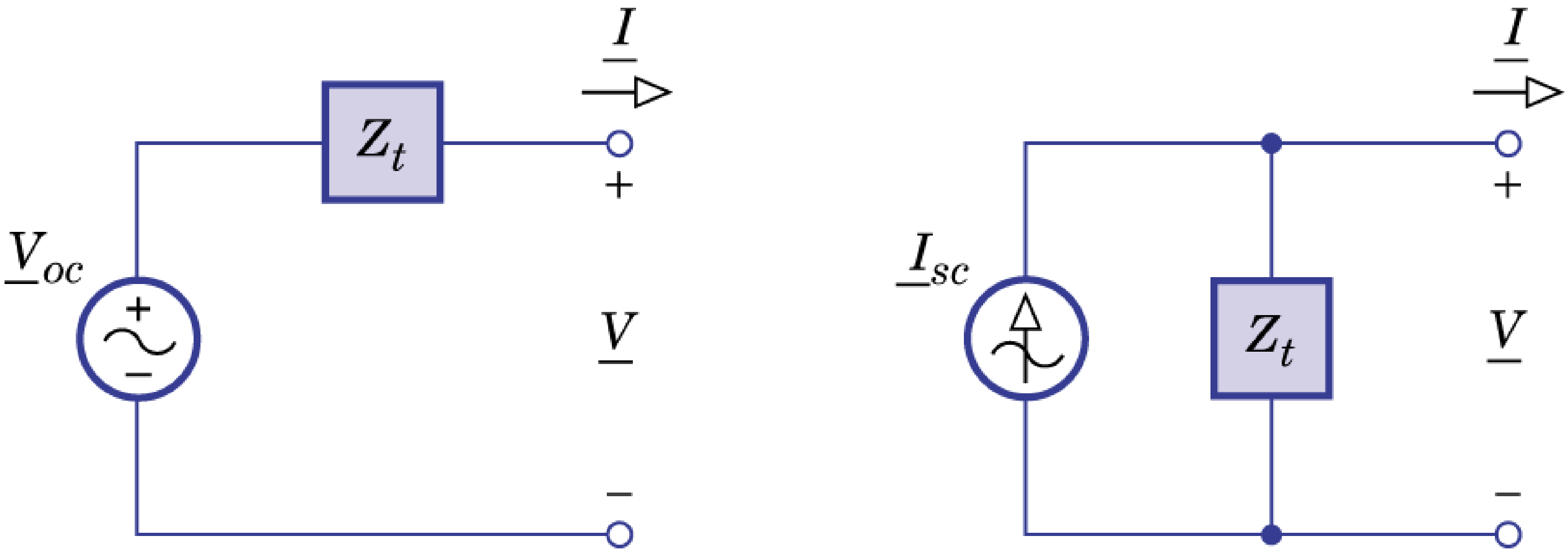
Superposition

- An ac source network is any two-terminal network that consists entirely of linear elements and sources. If there are more than one independent source, all of them must be at the same frequency so that the phasor method can be applied.

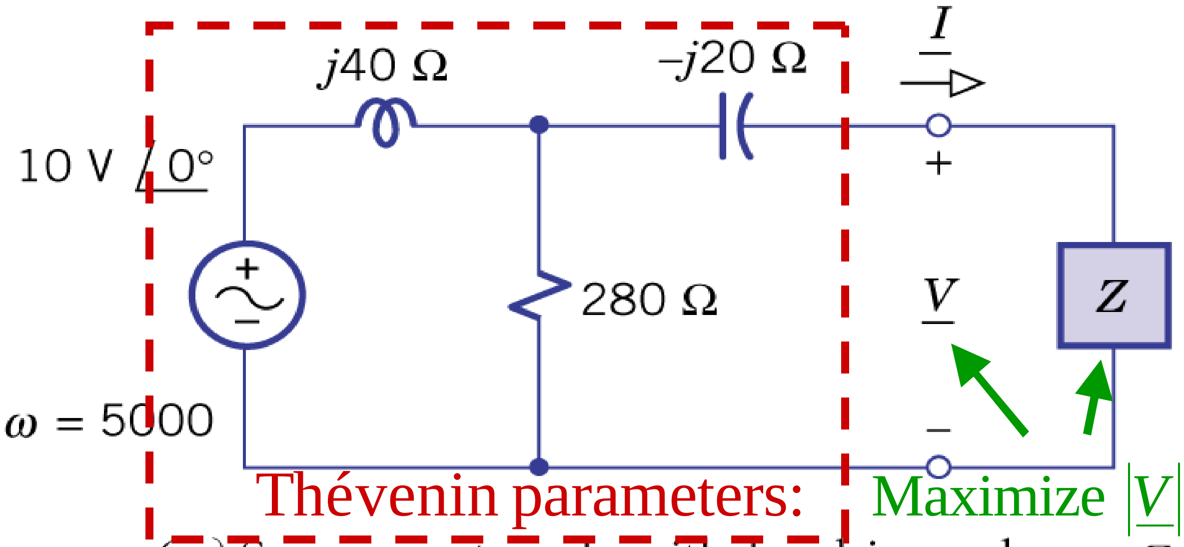
$\underline{V}_{oc}$

Frequency Domain Thévenin Parameters

- Frequency-domain Thevenin parameters:
  - the open-circuit voltage phasor: $\underline{V}_{oc}$
  - the short-circuit current phasor: $\underline{I}_{sc}$
  - Thévenin impedance: $Z_t = \underline{V}_{oc} / \underline{I}_{sc}$



Example 6.11: Application of an AC Norton Network.



Thévenin parameters:

Maximize $|\underline{V}|$

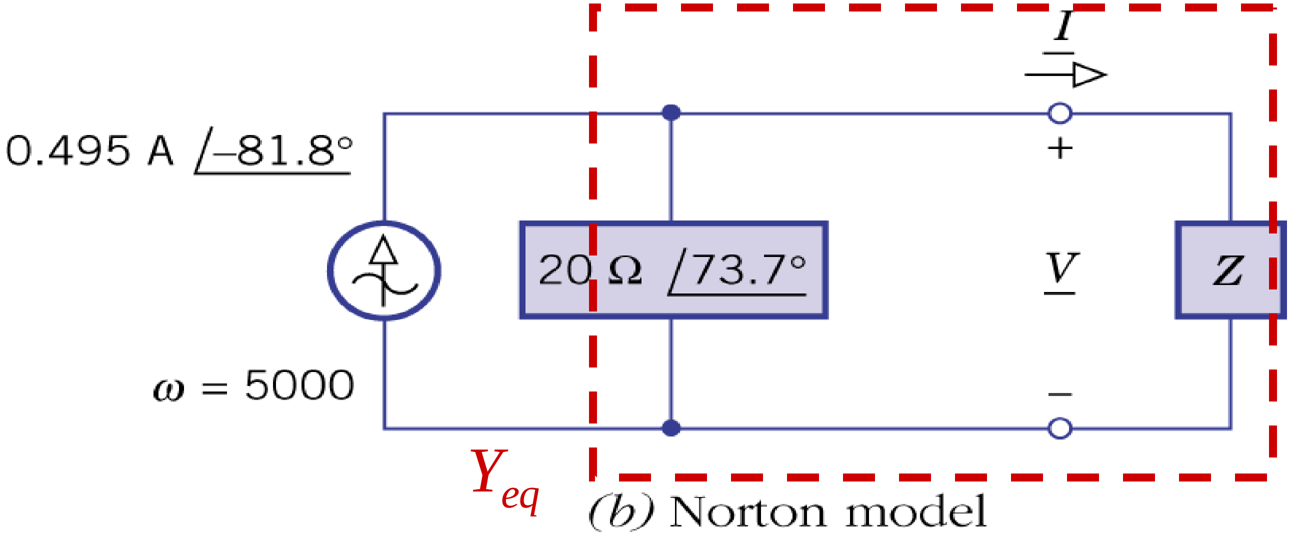
(a) Source network with load impedance Z

$$Z_t = j40 \| 280 - j20 = 20\Omega\angle 73.7^0$$

$$\underline{V}_{OC} = \frac{280}{280+j40}10 = 9.9V\angle -8.13^0$$

$$\underline{I}_{SC} = \frac{\underline{V}_{OC}}{Z_t} = 0.495A\angle -81.8^0$$

Example 6.11: (Cont.)



(b) Norton model

$$Y_{eq} = \frac{1}{Z_t} + \frac{1}{Z} \text{ and } \underline{V} = \underline{I}_{SC} / Y_{eq}$$

$|V|$ is maximum if $|Y_{eq}|$ is minimum

$$Y_{eq} = (0.014 + G) + j(B - 0.048)$$

$$Y = 0 + j0.048S$$

$$\underline{V} = 35.4V\angle -81.8^0$$

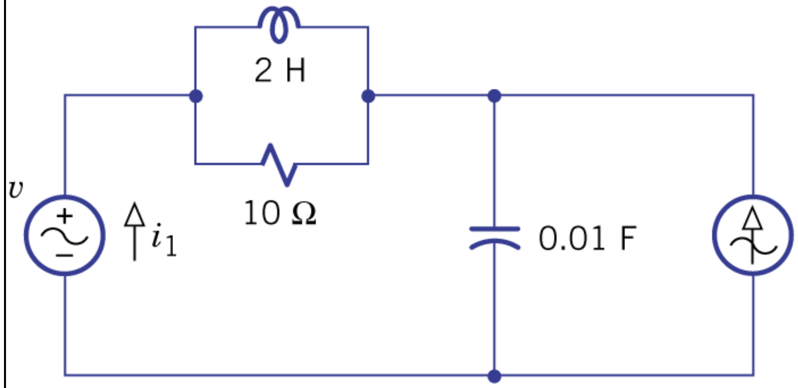
AC Mesh Analysis

- By using phasors, impedance and admittance, node analysis and mesh analysis are still applicable assuming all independent sources are at the same frequency.
- AC mesh analysis:

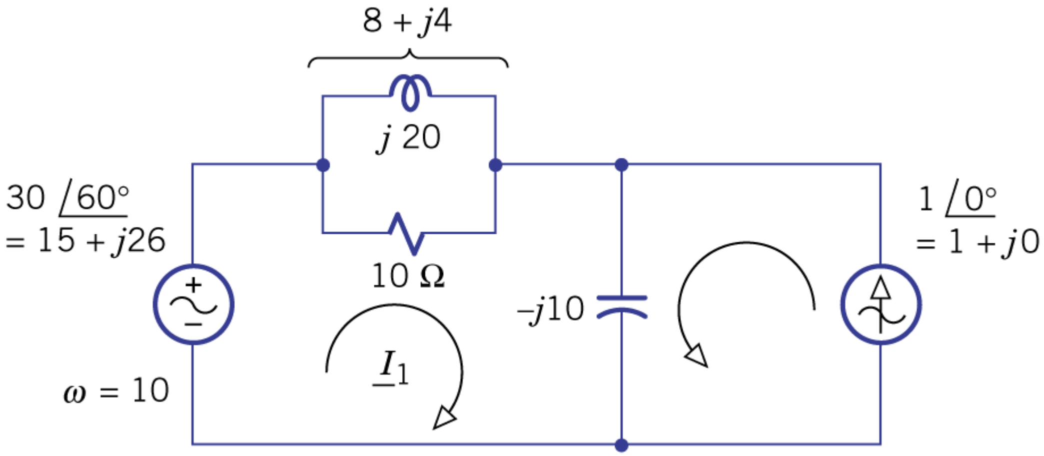
$$[Z][\underline{I}] = [\underline{V}_S] \quad \text{or} \quad [Z - \tilde{Z}][\underline{I}] = [\underline{\tilde{V}}_S]$$

with controlled sources

Example 6.12: Systematic AC Mesh Analysis



(*a*) Circuit with $v = 30 \cos(10t + 60°)$ V and $i = 1 \cos 10t$ A

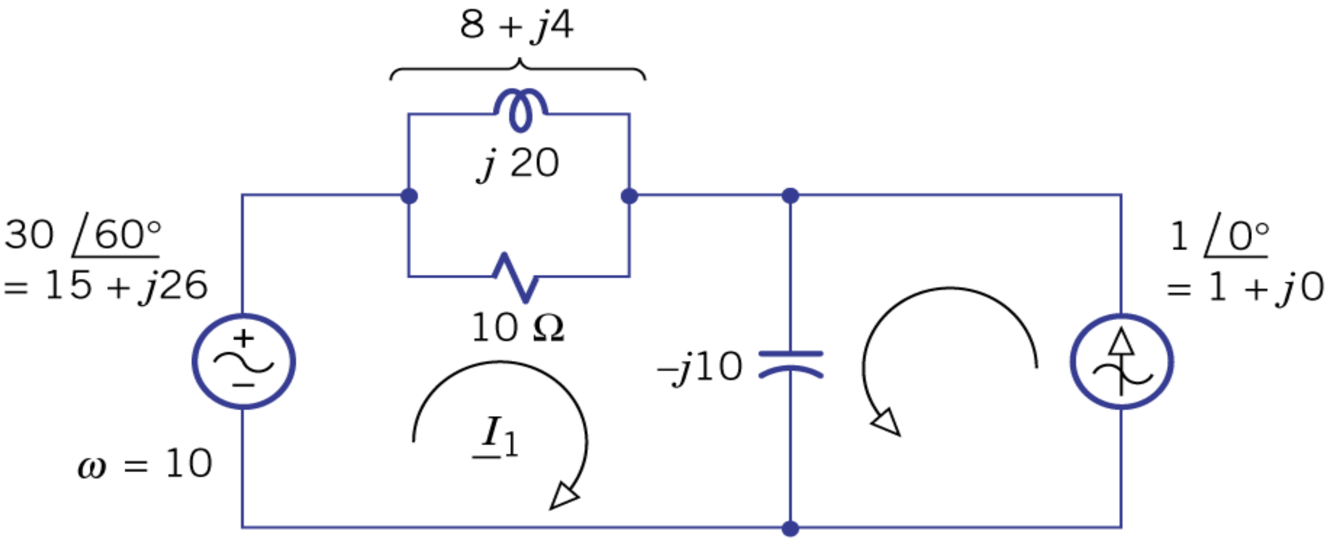


(*b*) Frequency-domain diagram with one unknown mesh current

Find i_1

Two sources at the same frequency

Example 6.12: (Cont.)



(b) Frequency-domain diagram with one unknown mesh current

Single mesh equation : $Z\underline{I}_1 = \underline{V}_S$

$Z = 8 + j4 - j10 = 8 - j6\Omega$

$\underline{V}_S = 15 + j26 - (-j10) = 15 + j36V$

$\underline{I}_1 = 3.9A\angle 104.3^0$

$i_1(t) = 3.9\cos(10t + 104.3^0)$

AC Node Analysis

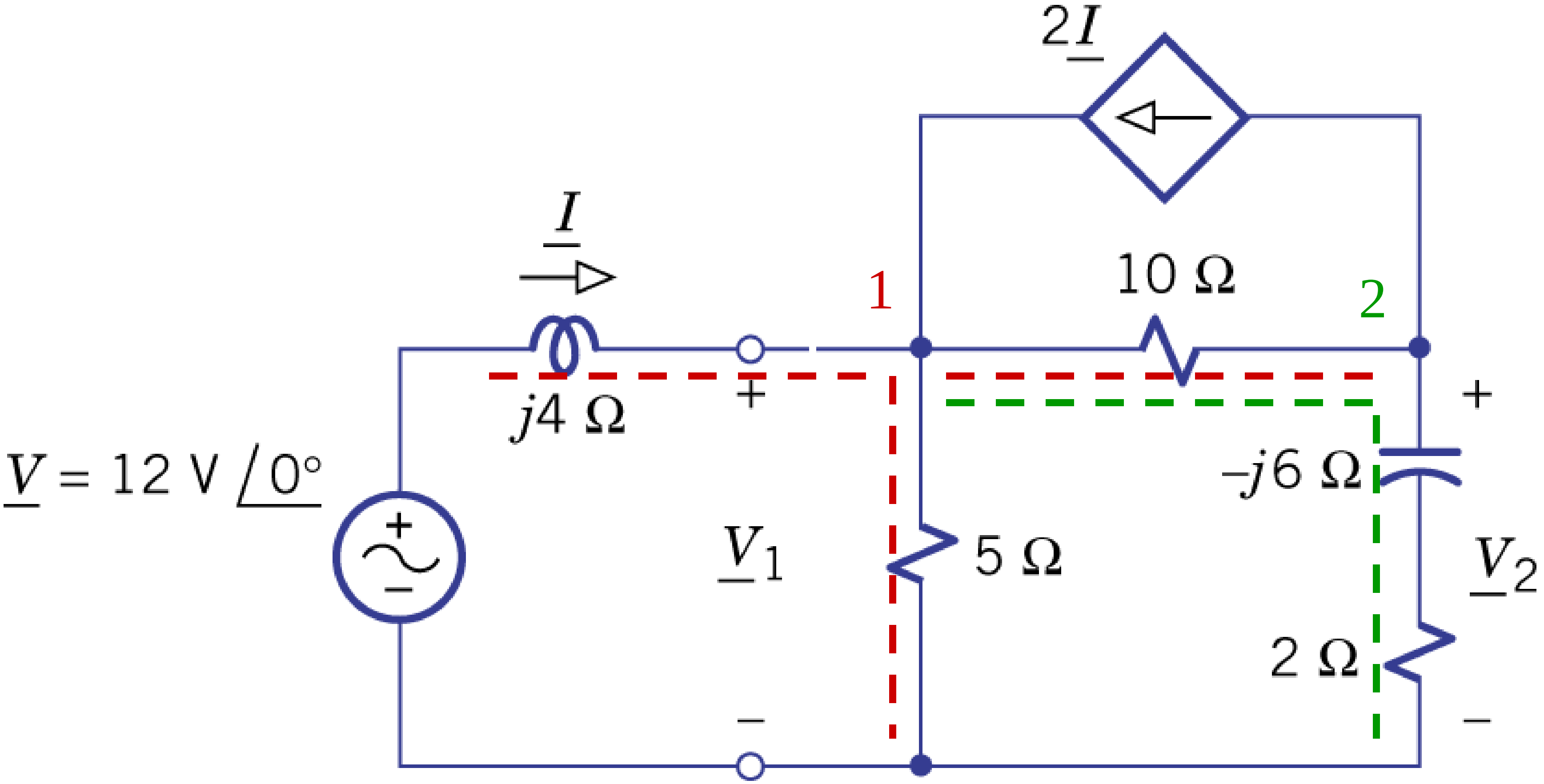
$$[Y][\underline{V}] = [\underline{I}_S]$$

or

$$[Y - \tilde{Y}][\underline{V}] = [\underline{\tilde{I}}_S]$$

with controlled sources

Example 6.13: Systematic AC Node Analysis



Constraint equation: $I=(\underline{V}-\underline{V}_1)/j4$

Example 6.13: (Cont.)

$$[Y]=\frac{1}{20}\begin{bmatrix}6-j5 & -2\\ -2 & 3+j3\end{bmatrix}$$

$$[\underline{I_s}]=\begin{bmatrix}2\underline{I}+\underline{V}/j4\\ -2\underline{I}\end{bmatrix}+\begin{bmatrix}-j9\\ j6\end{bmatrix}+\begin{bmatrix}j0.5 & 0\\ -j0.5 & 0\end{bmatrix}\begin{bmatrix}\underline{V}_1\\ \underline{V}_2\end{bmatrix}$$

$$\begin{bmatrix}6-j & -2\\ -2+j10 & 3+j3\end{bmatrix}\begin{bmatrix}\underline{V}_1\\ \underline{V}_2\end{bmatrix}=\begin{bmatrix}-j180\\ j20\end{bmatrix}$$

$$\underline{V}_1=10.4V\angle-22.3^0$$

$$\underline{I}=1.15A\angle-31.1^0$$

$$Z_1=\frac{\underline{V}_1}{\underline{I}}=9.03\Omega\angle8.8^0=8.92+j1.39\Omega$$

Chapter 6: Problem Set

- 7, 17, 24, 32, 36, 41, 44, 47, 51, 53, 57, 59